Sabbatical Mini-Report #4:
Metaphors, Mnemonics, and Learning Mathematics

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November 2006

This report attempts to connect two sets of concepts in a way that might help us understand our students' learning of mathematics: Metaphor (a linguistic, or verbal, construct) and Mnemonics (and related visual constructs).

Learning theory will be explored directly in its own report. However, for our purposes, accept the situation that learning theories might be very accurate in predicting the development of proficiency … while not explaining at all how the learning actually takes place. In other words, learning theory tends to be very descriptive of the behavior; the physical understanding of the brain is (although impressive) not even close to being complete.

METAPHORS

Some interesting results have been produced by researchers who look at learning mathematics by analyzing the metaphors used in attaching meaning to numbers and in performing procedures. One of the most common metaphors is "an equation is a balanced scale". Mathematically, this is not correct … equations are statements about quantities, and some of them are true ('balanced'). However, the metaphor is sometimes helpful. [A 'metaphor' is simply the equating of two unequal objects or ideas, in case you have forgotten the definition.]

In fact, Nunez and Lakoff (2005 and 2000 references) analyze a broad range of mathematical knowledge from a "metaphor" framework. Their conclusion from research is that learners usually develop their knowledge by successive stages of metaphors – and definitely not by logic.

This does not mean that all metaphors make good teaching devices. Researchers have found many weaknesses when metaphors are used … if they are used without proceeding to a more abstract view. (See Howson, page 26, in particular.) In the case of the "equation is balanced scale" metaphor, conflicts develop with subsequent skills and concepts – such as adding equations and 'squaring both sides'. The balance metaphor also interferes with function concepts.

Another common metaphor is "fraction is a division". As the result of this metaphor, students are frequently stuck on the image of a fraction as two numbers – they can't see $\frac{16}{12}$ as a number ("it's an operation, we've got to divide"). The metaphor totally breaks down with algebraic fractions most commonly used; consider $\frac{6x-12}{x^2+x-6}$ … we certainly can't divide! See the commentary by Smith (pages 94-95). In spite of these limitations, this "fraction is a division" summarizes many students' total understanding of a fraction; our teaching practices (in some cases) reinforces this metaphor and avoids a more abstract view of fractions. Abstraction is needed for the building of learning, and reduces the need for unlearning.

The last specific metaphor to be discussed here deals with infinity. We often say "infinity is the idea of the largest number"; mathematically, infinity is a number … and it's a very useful number. We can compare with infinity, and we can enumerate (count) with infinity; we just can't compute with infinity (not in the usual ways). See Lakoff (2000) pg 165 for more on the 'infinity metaphor'.

It appears that metaphors can not be avoided in learning mathematics, and that successful learning involves proceeding to more complete (accurate) metaphors.

MNEMONICS and VISUAL CONSTRUCTS

Another tool that we use to support our students is "mnemonics". The most common … "PEMDAS", as a way to remember the steps in the Order of Operations. There are actually many concerns about the use of mnemonics. Most importantly, it turns out that only some students benefit in the short term … they are not for all (or even most) students; see Bruning, pgs 72-73 for a look at the research. The basic finding: Students who have used mnemonics successfully can benefit from suggested mnemonics; other students are either not helped or the mnemonic actually interferes. [They also discuss other mnemonic types besides 'first letter'.]

The other concern about mnemonics comes from mathematicians and mathematics educators, and is very similar to the problems of metaphors: The mnemonic, like a metaphor, is not complete; the mnemonic is often further removed from the mathematics. Further learning can be difficult if a more abstract view does not follow the mnemonic. For example, the uses of the distributive property connect very well to the order of operations … but are only connected to PEMDAS by some explanation (unlearning). The order of operations is a way to connect meaning and properties; the mnemonic is a calculation tool.

The other visual construct seen in the literature is "visually salient rules". The classic example: $(4x^3)^2 = 16x^6$, where the power of a term works "just like you'd expect from looking at it". These visually salient rules are learned quickly and with a great deal of strength; the strength is so great that students apply them to visually similar problems: $(3x+5)^2 \neq 9x^2 + 25$. Many of the classic errors in algebra are rooted in non-visually salient rules. Another classic example (arithmetic and algebra): $\frac{3}{5} \bullet \frac{2}{7} = \frac{6}{35}$ (visually salient) versus $\frac{3}{5} + \frac{2}{7} \neq \frac{5}{12}$. See the article by Kirshner and Awtry for further discussion; they make the point that many errors are not the result of "mis-understanding" … they are a over-reliance on visual salience.

The critical difference between mnemonics and 'visually salient rules' is that mnemonics are a deliberate organizational tool (usually overly specific), while visually salient rules occur spontaneously due to human perception.

Many of our students will describe themselves as "visual learners"; that particular issue will be addressed separately (with the learning theories and other reports).

Let's summarize what I believe to be true about metaphors and visual constructs:

	Strength	Weakness	Other notes
Metaphor	Meaning connects with abstraction	Restrictive (not equal to abstraction)	Usually informal; not logical
Visual (mnemonic and 'salient')	Ease (accessible to learners)	Not directly connected with meaning or abstraction	"Unlearning" tends to be difficult; often transfers to the wrong situations
Abstraction	Supports further learning	Not as accessible	Often accessed by more complete metaphors

References:

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